$H-/H\infty$ robust fault detection observer for uncertain switched systems

Ahmad Farhat, Damien Koenig

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Abstract—**This paper addresses a method for robust fault detection (RFD) by maximizing the fault to residual sensitivity. It uses the newly developed $\mathcal{H}_-$ index properties and minimizing the well known $\mathcal{H}_\infty$ norm for worst case uncertainties and disturbance attenuation . These objectives are coupled to a transient response specification expressed by eigen region assignment formulation. The 3-objective robust fault detection problem is formulated as LMI feasibility problem in which a cost function is minimized subject to LMI constraints.**

Keywords : Residual generation, robust fault detection, uncertain switched systems, $\mathcal{H}_-$ index, $\mathcal{H}_\infty$ norm, LMI.

I. Introduction

Modern systems (vehicles, aircrafts...) are increasingly equipped with new mechanisms to improve safety. These new systems have often active parts using data from sensors. However, in case of malfunction of a sensor, the consequences can be dramatic. Early detection and diagnosis of process faults can help avoid abnormal event progression and improve reliability and safety issues [1].

Among numerous fault detection techniques (FD), model based design are one popular strategy that includes observer based approach, parity-space approach [2], eigenstructure assignment approach, parameter identification based methods [3]. The idea is to compute a residual signal by comparing the mathematical model of the plant and use the relations among several measured variables to extract information on possible changes caused by faults [4].

In practical applications, the residuals are corrupted by unknown inputs such as noises, disturbances, and uncertainties in the system model. Hence, the main objective of FD methods is to generate stable robust residuals that are insensitive to these noise and uncertainties, while sensitive to faults [5], [6].

Recent work on the $\mathcal{H}_-$ "norm" have been studied and various definition have been introduced [7]–[10]. It is the minimum "non-zero"singular value taken either at $\omega = 0$ [11], over a finite frequency range $[\underline{\omega}, \overline{\omega}]$, or over all frequency range $[0, \infty]$ [9].

The specifications and objectives under consideration include $\mathcal{H}_\infty$ performance, $\mathcal{H}_-$ performance and time domain constraints. The motivations for using this mixed performances are as follows:

*This work is supported by the French national project INOVE/ANR.
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- The $\mathcal{H}_\infty$ performance is useful to ensure the residual robustness to model uncertainties and disturbances.
- The $\mathcal{H}_-$ performance is convenient to express the residual sensitivity toward faults.
- The time domain constraint that is expressed by pole region assignment is useful to tune the transient response [12]–[14].

In this paper, an observer based filter is designed with the mixed $\mathcal{H}_-/\mathcal{H}_\infty$/eigenvalue assignment objectives. The desired observer is computed by solving a set of LMIs. A compromise between fault sensitivity, unknown input rejection, uncertainty robustness and eigen region assignment is optimized via a convex optimization algorithm.

The outline of this paper is as follows. After the Introduction, problem formulation is given in Section II. In section III, preliminaries for the synthesis of $\mathcal{H}_\infty$ observer, $\mathcal{H}_-$ fault detector. Robust fault detection observer scheme is given in Section IV using additive filter design. A min/max criterion is used to solve an optimization problem set by the LMIs. The above results are illustrated by a numerical example in Section V. Finally, Section VI shows the concluding remarks and the possible future work.

Notations: The notation used in this paper is standard. X^T is the transposed of matrix X, the star symbol $(\star)$ in a symmetric matrix denotes the transposed block in the symmetric position. The notation $P > (<)0$ means P is real symmetric positive (negative) definite matrix. 0 and I denote zeros and identity matrix of appropriate dimensions.

II. Problem Formulation

Consider the state space representation of the linear time uncertain switched system :

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \bar{A}_{\alpha(t)}x(t) + B_{\alpha(t)}u(t) \\ & & +E_{d,\alpha(t)}d(t) + E_{f,\alpha(t)}f(t) \\ y(t) & = & C_{\alpha(t)}x(t) + D_{\alpha(t)}u(t) \\ & & +F_{d,\alpha(t)}d(t) + F_{f,\alpha(t)}f(t) \end{array}\right. \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the measurement output vector, $u \in \mathbb{R}^m$ is the input vector, $d \in \mathbb{R}^{n_d}$ is the disturbance vector, $f \in \mathbb{R}^{n_f}$ is the vector of faults to be detected, $\alpha(t)$ is the switching signal, it is assumed known and measured.

Model uncertainties can be represented in different forms, in this study additive form is considered:

$$\bar{A}_{\alpha(t)} = A_{\alpha(t)} + \Delta_{x,\alpha(t)}N_{x,\alpha(t)} \tag{2}$$

The matrices A_α, B_α, $E_{d,\alpha}$, $E_{f,\alpha}$, C_α, D_α, $F_{d,\alpha}$ and $F_{f,\alpha}$ are the nominal LTI system matrices, they are known and in appropriate dimensions. $\Delta_{x,\alpha}$ is the state uncertainty matrix that is bounded $\|\Delta_{x,\alpha}\|_2 \leq \epsilon_{x,\alpha}$, $N_{x,\alpha}$ define the directions of these uncertainties.

In the following the subscript t is omitted without confusion for typing simplifications.

$$\left\{\begin{array}{lcl} \dot{x} &=& (A_\alpha + \Delta_{x,\alpha}N_{x,\alpha})x + B_\alpha u \\ & & +E_{d,\alpha}d + E_{f,\alpha}f \\ y &=& C_\alpha x + D_\alpha u + F_{d,\alpha}d + F_{f,\alpha}f \end{array}\right. \tag{3}$$

Introducing local variable χ, the system can be put in the form:

$$\left\{\begin{array}{lcl} \dot{x} &=& A_\alpha x + B_\alpha u + E_{d,\alpha}d + E_{f,\alpha}f + \chi \\ \chi &=& \Delta_{x,\alpha}N_{x,\alpha}x \\ y &=& C_\alpha x + D_\alpha u + F_{d,\alpha}d + F_{f,\alpha}f \end{array}\right. \tag{4}$$

The switched identity observer used by the residual generator is:

$$\left\{\begin{array}{lcl} \dot{\hat{x}} &=& A_\alpha \hat{x} + B_\alpha u + L_\alpha(y - \hat{y}) \\ \hat{y} &=& C_\alpha \hat{x} + D_\alpha u \\ r_\alpha &=& y - \hat{y} \end{array}\right. \tag{5}$$

Define the state error as $\tilde{x} = x - \hat{x}$. Then:

$$\begin{aligned} \dot{\hat{x}} &= A_\alpha \hat{x} + B_\alpha u + L_\alpha C_\alpha \tilde{x} \\ &\quad +L_\alpha F_{d,\alpha}d + L_\alpha F_{f,\alpha}f && (6) \\ \dot{\tilde{x}} &= A_\alpha x + \chi + B_\alpha u + E_{d,\alpha}d + E_{f,\alpha}f \\ &\quad -A_\alpha \hat{x} - B_\alpha u - L_\alpha C_\alpha \tilde{x} \\ &\quad -L_\alpha F_{d,\alpha}d - L_\alpha F_{f,\alpha}f \\ &= (A_\alpha - L_\alpha C_\alpha)\tilde{x} + (E_{d,\alpha} - L_\alpha F_{d,\alpha})d \\ &\quad +(E_{f,\alpha} - L_\alpha F_{f,\alpha})f + \chi && (7) \\ r_\alpha &= C_\alpha \tilde{x} + F_{d,\alpha}d + F_{f,\alpha}f && (8) \end{aligned}$$

Using the formulation in (4), it yields:

$$\left\{\begin{array}{lcl} \dot{\tilde{x}} &=& A^*_\alpha \tilde{x} + E^*_{d,\alpha}d + E^*_{f,\alpha}f + \chi \\ \chi &=& \Delta_{x,\alpha}N_{x,\alpha}x \\ r_\alpha &=& C_\alpha \tilde{x} + F_{d,\alpha}d + F_{f,\alpha}f \end{array}\right. \tag{9}$$

with $A^*_\alpha = A_\alpha - L_\alpha C_\alpha$, $E^*_{i,\alpha} = E_{i,\alpha} - L_\alpha F_{i,\alpha}$, $i \in \{f, d\}$.

Let the sensitivity functions of fault and disturbance to the residual be:

$$T_{rf_\alpha}(s) = C_\alpha(sI - A^*_\alpha)^{-1}E^*_{f,\alpha} + F_{f,\alpha} \tag{10}$$
$$T_{rd_\alpha}(s) = C_\alpha(sI - A^*_\alpha)^{-1}E^*_{d,\alpha} + F_{d,\alpha} \tag{11}$$

The objective of the $\mathcal{H}_-/\mathcal{H}_\infty$ switched FD observer is resumed by the following conditions:

$$\|T_{rd_\alpha}\|_\infty < \gamma_\alpha \tag{12}$$
$$\|T_{rf_\alpha}\|_- > \beta_\alpha \tag{13}$$

The problem is formulated as following: Find the matrix L_α that maximize β_α and minimize γ_α such that the switched FD observer is stable. The optimization criterion used in this paper is to maximize $\beta^2_\alpha - \gamma^2_\alpha$.

Assumption 1: In this study the pair (A_α, C_α) is assumed observable, or without loss of generality is detectable. It is a standard assumption for all fault detection problems.

III. PRELIMINARIES

Lemma 1: For any matrices X and Y with appropriate dimensions, the following statement holds:

$$X^TY + Y^TX < X^TX + Y^TY \tag{14}$$

Theorem 1: For a given uncertain switched system with faults as defined in (3), if there exits a symmetric matrix $P_\alpha > 0$ and positive scalars $\epsilon_{x,\alpha}$ and γ_α, such that the following inequality holds:

$$\begin{bmatrix} \Omega_{d,\alpha} & \Upsilon_{d,\alpha} & P_\alpha B_\alpha & -C^T_\alpha U^T_\alpha & P_\alpha \\ \star & J_{d,\alpha} & 0 & -F^T_{d,\alpha}U^T_\alpha & 0 \\ \star & \star & 0 & B^T_\alpha P_\alpha & 0 \\ \star & \star & \star & \Pi_\alpha & 0 \\ \star & \star & \star & \star & -\frac{1}{2}I \end{bmatrix} < 0 \tag{15}$$

where

$$\begin{aligned} \Omega_{d,\alpha} =& P_\alpha A_\alpha + U_\alpha C_\alpha + A^T_\alpha P_\alpha + \\ & C^T_\alpha U^T_\alpha + C^T_\alpha C_\alpha + 2\epsilon^2_x N^T_{x,\alpha}N_{x,\alpha}, \\ \Upsilon_{d,\alpha} =& P_\alpha E_{d,\alpha} + U_\alpha F_{d,\alpha} + C^T_\alpha F_{d,\alpha}, \\ J_{d,\alpha} =& F^T_{d,\alpha}F_{d,\alpha} - \gamma^2_\alpha I, \\ \Pi_\alpha =& A^T_\alpha P_\alpha + P_\alpha A_\alpha + 2\epsilon^2_x N^T_{x,\alpha}N_{x,\alpha} + C^T_\alpha C_\alpha \end{aligned}$$

Then a robust fault detection observer can be designed where the gain filter $L = -P^{-1}_\alpha U_\alpha$

Proof 1: The followings are the constraints for a general robust fault detection observer design:

- If there existe $P_\alpha > 0$, the sufficient stability condition considering the candidate Multiple Lyapunov Function (MLF):

$$V_\alpha = \tilde{x}^TP_\alpha\tilde{x} + \hat{x}^TP_\alpha\hat{x} \tag{16a}$$
$$\dot{V}_\alpha < 0 \tag{16b}$$

- For a positive scalar γ_α, the $\mathcal{H}_\infty$ disturbance rejection condition (11) is formulated as:

$$\|r_\alpha|_{f=0}\|_2 < \gamma_\alpha \|d\|_2 \tag{16c}$$

- The boundedness properties of the uncertainties are:

$$\begin{aligned} \chi^T\chi &= x^TN^T_{x,\alpha}\Delta^T_{x,\alpha}\Delta_{x,\alpha}N_{x,\alpha}x \\ &< \epsilon^2 x^TN^T_{x,\alpha}N_{x,\alpha}x \end{aligned} \tag{16d}$$

Using *Lemma 1*, we can write:

$$\begin{aligned} x^TN^T_{x,\alpha}N_{x,\alpha}x =& (\tilde{x} + \hat{x})^TN^T_{x,\alpha}N_{x,\alpha}(\tilde{x} + \hat{x}) \\ =& \tilde{x}^TN^T_{x,\alpha}N_{x,\alpha}\tilde{x} + \hat{x}^TN^T_{x,\alpha}N_{x,\alpha}\hat{x} \\ &+ \tilde{x}^TN^T_{x,\alpha}N_{x,\alpha}\hat{x} + \hat{x}^TN^T_{x,\alpha}N_{x,\alpha}\tilde{x} \\ <& \tilde{x}^TN^T_{x,\alpha}N_{x,\alpha}\tilde{x} + \hat{x}^TN^T_{x,\alpha}N_{x,\alpha}\hat{x} \\ &+ \tilde{x}^TN^T_{x,\alpha}N_{x,\alpha}\tilde{x} + \hat{x}^TN^T_{x,\alpha}N_{x,\alpha}\hat{x} \\ =& 2\tilde{x}^TN^T_{x,\alpha}N_{x,\alpha}\tilde{x} + 2\hat{x}^TN^T_{x,\alpha}N_{x,\alpha}\hat{x} \end{aligned} \tag{17}$$

Combining the equations (16a) - (16c) yields to:

$$\dot{V}_\alpha + r^T_\alpha r_\alpha - \gamma^2_\alpha d^Td < 0 \tag{18}$$

Let $V_\alpha = V_{1,\alpha} + V_{2,\alpha}$; $V_{1,\alpha} = \tilde{x}^TP_\alpha\tilde{x}$ and $V_{2,\alpha} = \hat{x}^TP_\alpha\hat{x}$.

Then using the properties (16g) and (17), the general form of the MLF derivatives are:

$$\begin{aligned}\dot{V}_{1,\alpha} =&(A_\alpha^*\tilde{x} + E_{d,\alpha}^*d + E_{f,\alpha}^*f + \chi)^T P_\alpha\tilde{x}\\ &+ \tilde{x}^T P_\alpha(A_\alpha^*\tilde{x} + E_{d,\alpha}^*d + E_{f,\alpha}^*f + \chi)\\ =&\tilde{x}^T(P_\alpha A_\alpha^* + A_\alpha^{*T}P_\alpha)\tilde{x} + \tilde{x}^T P_\alpha(E_{d,\alpha}^*d + E_{f,\alpha}^*f)\\ &+ (E_{d,\alpha}d + E_{f,\alpha}^*f)^T P_\alpha\tilde{x} + \tilde{x}^T P_\alpha\chi + \chi^T P_\alpha\tilde{x}\\ <&\tilde{x}^T(P_\alpha A_\alpha^* + A_\alpha^* P_\alpha)\tilde{x} + \tilde{x}^T P_\alpha(E_{d,\alpha}^*d + E_{f,\alpha}^*f)\\ &+ (E_{d,\alpha}^*d + E_{f,\alpha}^*f)^T P_\alpha\tilde{x} + \tilde{x}^T P_\alpha P_\alpha^T\tilde{x} + \chi^T\chi\\ <&\tilde{x}^T(P_\alpha A_\alpha^* + A_\alpha^* P_\alpha)\tilde{x} + \tilde{x}^T P_\alpha(E_{d,\alpha}^*d + E_{f,\alpha}^*f)\\ &+ (E_{d,\alpha}^*d + E_{f,\alpha}^*f)^T P_\alpha\tilde{x} + \tilde{x}^T P_\alpha^2\tilde{x} + \epsilon_x^2 x^T N_x^T N_x x\\ <&\tilde{x}^T(P_\alpha A_\alpha^* + A_\alpha^* P_\alpha)\tilde{x} + \tilde{x}^T P_\alpha(E_{d,\alpha}^*d + E_{f,\alpha}^*f)\\ &+ (E_{d,\alpha}^*d + E_{f,\alpha}^*f)^T P_\alpha\tilde{x} + \tilde{x}^T P_\alpha^2\tilde{x}\\ &+ \epsilon_x^2(2\tilde{x}^T N_{x,\alpha}^T N_{x,\alpha}\tilde{x} + 2\hat{x}^T N_{x,\alpha}^T N_{x,\alpha}\hat{x})\\ <&\tilde{x}^T(P_\alpha A_\alpha^* + A_\alpha^* P_\alpha + 2P_\alpha^2 + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha})\tilde{x}\\ &+ \tilde{x}^T P_\alpha E_{d,\alpha}^* d + d^T E_{d,\alpha}^{*T} P_\alpha\tilde{x}\\ &+ \tilde{x}^T P_\alpha E_{f,\alpha}^* f + f^T E_{f,\alpha}^{*T} P_\alpha\tilde{x}\\ &+ 2\epsilon_x^2\hat{x}^T N_{x,\alpha}^T N_{x,\alpha}\hat{x}\end{aligned} \tag{19}$$

$$\begin{aligned}\dot{V}_{2,\alpha} =&\dot{\hat{x}}^T P_\alpha\hat{x} + \hat{x}^T P_\alpha\dot{\hat{x}}\\ =&(A_\alpha\hat{x} + B_\alpha u + L_\alpha C_\alpha\tilde{x} + L_\alpha F_{d,\alpha}d\\ &+ L_\alpha F_{f,\alpha}f)^T P_\alpha\hat{x} + \hat{x}^T P_\alpha(A_\alpha\hat{x} + B_\alpha u\\ &+ L_\alpha C_\alpha\tilde{x} + L_\alpha F_{d,\alpha}d + L_\alpha F_{f,\alpha}f)\\ =&\hat{x}^T(A_\alpha^T P_\alpha + P_\alpha A_\alpha)\hat{x} + \hat{x}^T P_\alpha B_\alpha u + u^T B_\alpha^T P_\alpha\hat{x}\\ &+ \hat{x}^T P_\alpha L_\alpha C_\alpha\tilde{x} + \tilde{x}^T C_\alpha^T L_\alpha^T P_\alpha\hat{x} + \hat{x}^T P_\alpha L_\alpha F_{d,\alpha}d\\ &+ d^T F_{d,\alpha}^T L_\alpha^T P_\alpha\hat{x} + \hat{x}^T P_\alpha L_\alpha F_{f,\alpha}f\\ &+ f^T F_{f,\alpha}^T L_\alpha^T P_\alpha\hat{x}\end{aligned} \tag{20}$$

In the fault free case, the inequalities (18)-(20) yield to:

$$\begin{aligned}&\dot{V}_\alpha|_{f=0} + r_\alpha^T r_\alpha - \gamma_\alpha^2 d^T d\\ <&\tilde{x}^T(P_\alpha A_\alpha^* + A_\alpha^* P_\alpha + 2P_\alpha^2 + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha})\tilde{x}\\ &+ \tilde{x}^T P_\alpha E_{d,\alpha}^* d + d^T E_{d,\alpha}^{*T} P_\alpha\tilde{x} + 2\epsilon_x^2\tilde{x}^T N_{x,\alpha}^T N_{x,\alpha}\hat{x}\\ &+ \hat{x}^T(A_\alpha^T P_\alpha + P_\alpha A_\alpha)\hat{x} + \hat{x}^T P_\alpha B_\alpha u\\ &+ u^T B_\alpha^T P_\alpha\hat{x} + \hat{x}^T P_\alpha L_\alpha C_\alpha\tilde{x} + \tilde{x}^T C_\alpha^T L_\alpha^T P_\alpha\hat{x}\\ &+ \tilde{x}^T C_\alpha^T F_{d,\alpha}d + d^T F_{d,\alpha}^T C_\alpha\tilde{x} + d^T F_{d,\alpha}^T F_{d,\alpha}d\\ &+ \hat{x}^T P_\alpha L_\alpha F_{d,\alpha}d + d^T F_{d,\alpha}^T L_\alpha^T P_\alpha\hat{x}\\ &+ \tilde{x}^T C_\alpha^T C_\alpha\tilde{x} - \gamma_\alpha^2 d^T d\\ <&\tilde{x}^T(P_\alpha A_\alpha^* + A_\alpha^* P_\alpha + 2P_\alpha^2 + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha} + C_\alpha^T C_\alpha)\tilde{x}\\ &+ \tilde{x}^T(P_\alpha E_{d,\alpha}^* + C_\alpha^T F_{d,\alpha})d + d^T(E_{d,\alpha}^{*T} P_\alpha + F_{d,\alpha}^T C_\alpha)\tilde{x}\\ &+ \hat{x}^T(A_\alpha^T P_\alpha + P_\alpha A_\alpha + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha})\hat{x} + \hat{x}^T P_\alpha B_\alpha u\\ &+ u^T B_\alpha^T P_\alpha\hat{x} + \hat{x}^T P_\alpha L_\alpha C_\alpha\tilde{x} + \tilde{x}^T C_\alpha^T L_\alpha^T P_\alpha\hat{x}\\ &+ \hat{x}^T P_\alpha L_\alpha F_{d,\alpha}d + d^T F_{d,\alpha}^T L_\alpha^T P_\alpha\hat{x}\\ &+ d^T(F_{d,\alpha}^T F_{d,\alpha} - \gamma_\alpha^2 I)d = \Gamma_\alpha|_{f=0} < 0\end{aligned} \tag{21}$$

Solving the set of inequalities $\Gamma_\alpha|_{f=0} < 0$ guarantees the solution for (18).

In the quadratic form:

$$\begin{bmatrix}\tilde{x}\\ d\\ u\\ \hat{x}\end{bmatrix}^T \begin{bmatrix}\Omega_{d,\alpha}^* & \Upsilon_{d,\alpha}^* & P_\alpha B_\alpha & C_\alpha^T L_\alpha^T P_\alpha\\ \star & J_{d,\alpha} & 0 & F_{d,\alpha}^T L_\alpha^T P_\alpha\\ \star & \star & 0 & B_\alpha^T P_\alpha\\ \star & \star & \star & \Pi_\alpha\end{bmatrix} \begin{bmatrix}\tilde{x}\\ d\\ u\\ \hat{x}\end{bmatrix} < 0 \tag{22}$$

where

$$\begin{aligned}\Omega_{d,\alpha}^* &= P_\alpha A_\alpha^* + A_\alpha^* P_\alpha + 2P_\alpha^2 + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha} + C_\alpha^T C_\alpha,\\ \Upsilon_{d,\alpha}^* &= P_\alpha E_{d,\alpha}^* + C_\alpha^T F_{d,\alpha},\\ J_{d,\alpha} &= F_{d,\alpha}^T F_{d,\alpha} - \gamma_\alpha^2 I,\\ \Pi_\alpha &= A_\alpha^T P_\alpha + P_\alpha A_\alpha + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha} + C_\alpha^T C_\alpha\end{aligned}$$

This inequality holds $\forall \begin{bmatrix}\tilde{x}^T & d^T & u^T & \hat{x}^T\end{bmatrix}^T \neq 0$, thus:

$$\begin{bmatrix}\Omega_{d,\alpha}^* & \Upsilon_{d,\alpha}^* & P_\alpha B_\alpha & C_\alpha^T L_\alpha^T P_\alpha\\ \star & J_{d,\alpha}^* & 0 & F_{d,\alpha}^T L_\alpha^T P_\alpha\\ \star & \star & 0 & B_\alpha^T P_\alpha\\ \star & \star & \star & \Pi_\alpha\end{bmatrix} < 0 \tag{23}$$

This BMI is transformed into LMI by replacing $U_\alpha = -P_\alpha L_\alpha$, and using Schur complement formula for $P_\alpha^T P_\alpha$. It follows:

$$\begin{bmatrix}\Omega_{d,\alpha} & \Upsilon_{d,\alpha} & P_\alpha B_\alpha & -C_\alpha^T U_\alpha^T & P_\alpha\\ \star & J_{d,\alpha} & 0 & -F_{d,\alpha}^T U_\alpha^T & 0\\ \star & \star & 0 & B_\alpha^T P_\alpha & 0\\ \star & \star & \star & \Pi_\alpha & 0\\ \star & \star & \star & \star & -\frac{1}{2}I\end{bmatrix} < 0 \tag{24}$$

$$\begin{aligned}\Omega_{d,\alpha} =&P_\alpha A_\alpha + U_\alpha C_\alpha + A_\alpha^T P_\alpha +\\ &C_\alpha^T U_\alpha^T + C_\alpha^T C_\alpha + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha}\\ \Upsilon_{d,\alpha} =&P_\alpha E_{d,\alpha} + U_\alpha F_{d,\alpha} + C_\alpha^T F_{d,\alpha}\end{aligned}$$ □

Remark 1: The $\hat{x}^T P_\alpha \hat{x}$ term has been added to the MLF in (16a), in order to ensure the feasibility of the LMI (15). In fact, without this term, the diagonal term Π_α would be equal to $2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha}$ which is positive, that leads to an infeasible LMI.

Theorem 2: For a given uncertain switched system with faults as defined in (3), if there exits a symmetric matrix $P_\alpha > 0$ and positive scalars $\epsilon_{x,\alpha}$ and β_α, such that the following inequality holds:

$$\begin{bmatrix}\Omega_{f,\alpha} & \Upsilon_{f,\alpha} & P_\alpha B_\alpha & -C_\alpha^T U_\alpha^T & P_\alpha\\ \star & J_{f,\alpha} & 0 & F_{f,\alpha}^T U_\alpha^T & 0\\ \star & \star & 0 & B_\alpha^T P_\alpha & 0\\ \star & \star & \star & \Pi_\alpha & 0\\ \star & \star & \star & \star & -\frac{1}{2}I\end{bmatrix} < 0 \tag{25}$$

where

$$\begin{aligned}\Omega_{f,\alpha} =&P_\alpha A_\alpha + U_\alpha C_\alpha + A_\alpha^T P_\alpha +\\ &C_\alpha^T U_\alpha^T - C_\alpha^T C_\alpha + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha},\\ \Upsilon_{f,\alpha} =&P_\alpha E_{f,\alpha} + U_\alpha F_{f,\alpha} + C_\alpha^T F_{f,\alpha},\\ J_{d,\alpha} =&- F_{f,\alpha}^T F_{f,\alpha} + \beta_\alpha^2 I,\\ \Pi_\alpha =&A_\alpha^T P_\alpha + P_\alpha A_\alpha + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha} + C_\alpha^T C_\alpha\end{aligned}$$

Then a robust fault detection observer can be designed where the gain filter $L_\alpha = -P_\alpha^{-1} U_\alpha$. The residual to fault sensitivty is at least β_α.

Proof 2: The proof of this theorem is very similar to *Theorem 1*, with some some terms and sign differences.

The followings are the constraints for a general robust fault detection observer design:

- For $\beta_\alpha > 0$, the fault sensitivity condition (10) or the $\mathcal{H}_-$ index, is formulated as:

$$\left\| r_\alpha|_{d=0} \right\|_2 > \beta_\alpha \left\| f \right\|_2 \tag{26}$$

- The stability and the boundedness properties of the uncertainties are the same in *Theorem 1*.

The inequaltity to solve is:

$$\dot{V}_\alpha|_{d=0} - r_\alpha^T r_\alpha + \beta_\alpha^2 f^T f < 0 \tag{27}$$

In the disturbance free case, the inequalities (19)-(20) and (27) yield to:

$$\begin{aligned}
&\dot{V}_\alpha|_{f=0} - r_\alpha^T r_\alpha + \beta_\alpha^2 f^T f \\
<&\tilde{x}^T (P_\alpha A_\alpha^* + A_\alpha^* P_\alpha + 2P_\alpha^2 + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha}) \tilde{x} \\
&+ \tilde{x}^T P_\alpha E_{f,\alpha}^* f + f^T E_{f,\alpha}^{*T} P_\alpha \tilde{x} + 2\epsilon_x^2 \hat{x}^T N_{x,\alpha}^T N_{x,\alpha} \hat{x} \\
&+ \hat{x}^T (A_\alpha^T P_\alpha + P_\alpha A_\alpha) \hat{x} + \hat{x}^T P_\alpha B_\alpha u \\
&+ u^T B_\alpha^T P_\alpha \hat{x} + \hat{x}^T P_\alpha L_\alpha C_\alpha \tilde{x} + \tilde{x}^T C_\alpha^T L_\alpha^T P_\alpha \hat{x} \\
&- \tilde{x}^T C_\alpha^T F_{f,\alpha} f - f^T F_{f,\alpha}^T C_\alpha \tilde{x} - f^T F_{f,\alpha}^T F_{f,\alpha} f \\
&- \hat{x}^T P_\alpha L_\alpha F_{f,\alpha} f - f^T F_{f,\alpha}^T L_\alpha^T P_\alpha \hat{x} \\
&- \tilde{x}^T C_\alpha^T C_\alpha \tilde{x} + \beta_\alpha^2 f^T f \\
<&\tilde{x}^T (P_\alpha A_\alpha^* + A_\alpha^* P_\alpha + 2P_\alpha^2 + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha} - C_\alpha^T C_\alpha) \tilde{x} \\
&+ \tilde{x}^T (P_\alpha E_{f,\alpha}^* - C_\alpha^T F_{f,\alpha}) f + f^T (E_{f,\alpha}^{*T} P_\alpha - F_{f,\alpha}^T C_\alpha) \tilde{x} \\
&+ \hat{x}^T (A_\alpha^T P_\alpha + P_\alpha A_\alpha + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha}) \hat{x} + \hat{x}^T P_\alpha B_\alpha u \\
&+ u^T B_\alpha^T P_\alpha \hat{x} + \hat{x}^T P_\alpha L_\alpha C_\alpha \tilde{x} + \tilde{x}^T C_\alpha^T L_\alpha^T P_\alpha \hat{x} \\
&- \hat{x}^T P_\alpha L_\alpha F_{f,\alpha} f - f^T F_{f,\alpha}^T L_\alpha^T P_\alpha \hat{x} \\
&+ f^T (-F_{f,\alpha}^T F_{f,\alpha} + \beta_\alpha^2 I) f = \Gamma_\alpha|_{d=0} < 0
\end{aligned} \tag{28}$$

Solving the set of inequalities $\Gamma_\alpha|_{d=0} < 0$ guarantees the solution for (27). In the quadratic form:

$$\begin{bmatrix} \tilde{x} \\ f \\ u \\ \hat{x} \end{bmatrix}^T \begin{bmatrix} \Omega_{f,\alpha}^* & \Upsilon_{f,\alpha}^* & P_\alpha B_\alpha & C_\alpha^T L_\alpha^T P_\alpha \\ \star & J_{f,\alpha} & 0 & -F_{f,\alpha}^T L_\alpha^T P_\alpha \\ \star & \star & 0 & B_\alpha^T P_\alpha \\ \star & \star & \star & \Pi_\alpha \end{bmatrix} \begin{bmatrix} \tilde{x} \\ f \\ u \\ \hat{x} \end{bmatrix} < 0 \tag{29}$$

where

$$\begin{aligned}
\Omega_{f,\alpha}^* &= P_\alpha A_\alpha^* + A_\alpha^* P_\alpha + 2P_\alpha^2 + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha} - C_\alpha^T C_\alpha, \\
\Upsilon_{f,\alpha}^* &= P_\alpha E_{f,\alpha}^* - C_\alpha^T F_{f,\alpha}, \\
J_{f,\alpha} &= -F_{f,\alpha}^T F_{f,\alpha} + \beta_\alpha^2 I, \\
\Pi_\alpha &= A_\alpha^T P_\alpha + P_\alpha A_\alpha + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha} + C_\alpha^T C_\alpha
\end{aligned}$$

This inequality holds $\forall \begin{bmatrix} \tilde{x}^T & f^T & u^T & \hat{x}^T \end{bmatrix}^T \neq 0$, thus:

$$\begin{bmatrix} \Omega_{f,\alpha}^* & \Upsilon_{f,\alpha}^* & P_\alpha B_\alpha & C_\alpha^T L_\alpha^T P_\alpha \\ \star & J_{f,\alpha}^* & 0 & -F_{f,\alpha}^T L_\alpha^T P_\alpha \\ \star & \star & 0 & B_\alpha^T P_\alpha \\ \star & \star & \star & \Pi_\alpha \end{bmatrix} < 0 \tag{30}$$

This BMI is transformed into LMI by replacing $U_\alpha = -P_\alpha L_\alpha$, and using Schur complement formula for $P_\alpha^T P_\alpha$. It follows:

$$\begin{bmatrix} \Omega_{f,\alpha} & \Upsilon_{f,\alpha} & P_\alpha B_\alpha & -C_\alpha^T U_\alpha^T & P_\alpha \\ \star & J_{f,\alpha} & 0 & F_{f,\alpha}^T U_\alpha^T & 0 \\ \star & \star & 0 & B_\alpha^T P_\alpha & 0 \\ \star & \star & \star & \Pi_\alpha & 0 \\ \star & \star & \star & \star & -\frac{1}{2} I \end{bmatrix} < 0 \tag{31}$$

$$\begin{aligned}
\Omega_{f,\alpha} =& P_\alpha A_\alpha + U_\alpha C_\alpha + A_\alpha^T P_\alpha + \\
& C_\alpha^T U_\alpha^T - C_\alpha^T C_\alpha + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha} \\
\Upsilon_{f,\alpha} =& P_\alpha E_{f,\alpha} + U_\alpha F_{f,\alpha} - C_\alpha^T F_{f,\alpha}
\end{aligned}$$

□

Remark 2: As it has been demonstrated in [9], P_α in the LMI for the $\mathcal{H}_-$ observer is not required to be sign definite, and this condition does not ensure the stability of the observer. However, joint $\mathcal{H}_-/\mathcal{H}_\infty$ observer is stable since it is guaranteed by *Theorem 1*: P_α is the same matrix in the LMI formulation, its sign definitiveness is thus imposed.

Theorem 3: For a given square $n \times n$ matrix A_α, if there exits a symmetric matrix $P_\alpha > 0$ and a positive scalar ξ_α such that the following inequality holds:

$$A_\alpha^T P_\alpha + P_\alpha A_\alpha - 2\xi_\alpha P_\alpha < 0 \tag{32}$$

Then all eigenvalues of A_α are on left plane of ξ_α.

Proof 3: (32) is a result of a classical Lyapunov function for sufficient condition of stability.

The system $\dot{x} = (A_\alpha - \xi_\alpha I)x$ is stable if there exist a symmetric matrix $P_\alpha > 0$ where $V = x^T P_\alpha x$, $\dot{V} < 0$.

Thus:

$$(A_\alpha - \xi_\alpha I)^T P_\alpha + P_\alpha (A_\alpha - \xi_\alpha I) < 0 \tag{33}$$

which is equivalent to (32).

□

IV. ROBUST FAULT DETECTION OBSERVER DESIGN

The 3-objectives of the observer are: (a) robustness against perturbation and uncertainties, (b) sensitivity toward faults and (c) a correct time response for fault detection. In order to meet all these constraint, the developed design in this section can be adopted.

Whilst the raise time constant is predefined and is inversely proportional to ξ_α, the coefficients γ_α and β_α have to be minimized/maximized respectively. This consists in solving a set of LMIs as an optimization problem where the criterion to minimize is $\gamma_\alpha^2 - \beta_\alpha^2$.

These LMIs are:

$$\begin{bmatrix} \Omega_{d,\alpha} & \Upsilon_{d,\alpha} & P_\alpha B_\alpha & -C_\alpha^T U_\alpha^T & P_\alpha \\ \star & J_{d,\alpha} & 0 & -F_{d,\alpha}^T U_\alpha^T & 0 \\ \star & \star & 0 & B_\alpha^T P_\alpha & 0 \\ \star & \star & \star & \Pi_\alpha & 0 \\ \star & \star & \star & \star & -\frac{1}{2}I \end{bmatrix} < 0 \quad (34)$$

$$\begin{bmatrix} \Omega_{f,\alpha} & \Upsilon_{f,\alpha} & P_\alpha B_\alpha & -C_\alpha^T U_\alpha^T & P_\alpha \\ \star & J_{f,\alpha} & 0 & F_{f,\alpha}^T U_\alpha^T & 0 \\ \star & \star & 0 & B_\alpha^T P_\alpha & 0 \\ \star & \star & \star & \Pi_\alpha & 0 \\ \star & \star & \star & \star & -\frac{1}{2}I \end{bmatrix} < 0 \quad (35)$$

$$P_\alpha A_\alpha + A_\alpha^T P_\alpha + U_\alpha C_\alpha + C_\alpha^T U_\alpha^T - 2\xi_\alpha P_\alpha < 0 \quad (36)$$

$$P_\alpha > 0 \quad (37)$$

with

$$\begin{aligned} \Omega_{d,\alpha} &= P_\alpha A_\alpha + U_\alpha C_\alpha + A_\alpha^T P_\alpha + C_\alpha^T U_\alpha^T + C_\alpha^T C_\alpha \\ \Omega_{f,\alpha} &= P_\alpha A_\alpha + U_\alpha C_\alpha + A_\alpha^T P_\alpha + C_\alpha^T U_\alpha^T - C_\alpha^T C_\alpha \\ \Upsilon_{d,\alpha} &= P_\alpha E_{d,\alpha} + U_\alpha F_{d,\alpha} + C_\alpha^T F_{d,\alpha} \\ \Upsilon_{f,\alpha} &= P_\alpha E_{f,\alpha} + U_\alpha F_{f,\alpha} - C_\alpha^T F_{f,\alpha} \\ J_{d,\alpha} &= F_{d,\alpha}^T F_{d,\alpha} - \gamma_\alpha^2 I \\ J_{f,\alpha} &= -F_{f,\alpha}^T F_{f,\alpha} + \beta_\alpha^2 I \\ \Pi_\alpha &= A_\alpha^T P_\alpha + P_\alpha A_\alpha + 2\epsilon_x^2 N_{x,\alpha}^T N_{x,\alpha} + C_\alpha^T C_\alpha \end{aligned}$$

The gain filter is $L_\alpha = -P_\alpha^{-1} U_\alpha$.

Using Matlab optimization tools such YALMIP or SeDuMi, the set of LMIs is then solved with the $\gamma_\alpha^2 - \beta_\alpha^2$ criterion minimization.

The designed observer from (5) can be finally put in the following form:

$$\begin{cases} \dot{\hat{x}} &= (A_\alpha - L_\alpha C_\alpha)\hat{x} + \begin{bmatrix}(B_\alpha - L_\alpha D_\alpha) & L_\alpha\end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \\ r &= -C_\alpha \hat{x} + \begin{bmatrix} -D_\alpha & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \end{cases} \quad (38)$$

To improve the stability and the switching rule, we have considered a weighting function between two successive switched observers. it has the form:

$$r_\alpha = (1-p)r_\alpha + (p)r_{\alpha+1} \quad (39)$$

where $p \in [0, 1]$.

V. Example

Consider the problem of the FD in the lateral control of a vehicle. Some experimental data have been taked from a real "Renaul Scenic", provided by the french laboratory MIPS.

The widely used bicycle-model is a good representation of the system [15]. However, this model is non-linear since it has $\frac{1}{v}$ and $\frac{1}{v^2}$ terms in it:

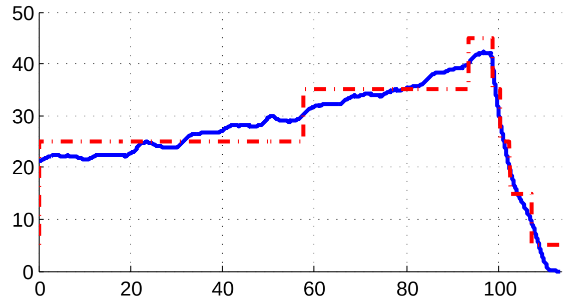

Fig. 1. Longitudinal velocity [km/h] and switching rule

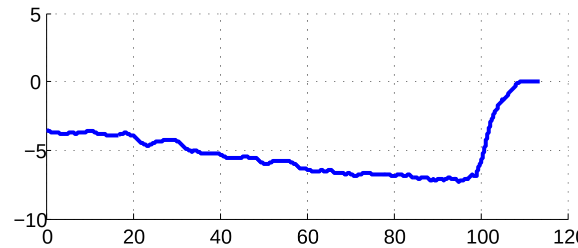

Fig. 2. Lateral acceleration [m/s^2]

$$\begin{bmatrix} \dot{\beta}(t) \\ \ddot{\psi}(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_r + c_f}{mv(t)} & \frac{c_r l_r - c_f l_f}{mv^2(t)} - 1 \\ \frac{c_r l_r - c_f l_f}{I_z} & -\frac{c_r l_r^2 + c_f l_f^2}{I_z v(t)} \end{bmatrix} \begin{bmatrix} \beta(t) \\ \dot{\psi}(t) \end{bmatrix} + \begin{bmatrix} \frac{c_f}{m} \\ \frac{c_r l_f}{I_z} \end{bmatrix} u_L(t) + \begin{bmatrix} \frac{1}{m} \\ \frac{l_w}{I_z} \end{bmatrix} F_w(t) \quad (40)$$

$$y = \begin{bmatrix} \frac{c_r + c_f}{m} & \frac{c_f l_f - c_r l_r}{m} \end{bmatrix} \begin{bmatrix} \beta(t) \\ \dot{\psi}(t) \end{bmatrix} + \frac{c_f}{m} u_L(t) \quad (41)$$

The measured output is the lateral acceleration γ_L, the entry command is the steering angle u_L, the sates are the side slip angle β and the yaw rate $\dot{\psi}$, and we consider the wind force as an unknown perturbation signal F_w.

In this approach, the system is linearized around multiple points v_α as shown in dashed curve of figure 1. The switching signal $\alpha(t)$ is calculated as the integer part of the output of the division: $\frac{v(t)}{\delta}$. v_α is defined as $v_\alpha = \delta\alpha(t)$.

Using a Taylor expansion around the points v_α:

$$\frac{1}{v}|_{v=v_\alpha} = \frac{1}{v_\alpha} - \frac{1}{v_\alpha^2}(v - v_\alpha) + \mathcal{O}(\frac{1}{v^2}) \quad (42)$$

$$\frac{1}{v^2}|_{v=v_\alpha} = \frac{1}{v_\alpha^2} - \frac{2}{v_\alpha^3}(v - v_\alpha) + \mathcal{O}(\frac{1}{v^3}) \quad (43)$$

Then

$$A = \underbrace{A_0 + \frac{1}{v_\alpha}A_1 + \frac{1}{v_\alpha^2}A_2}_{A_\alpha} + \underbrace{(-\frac{1}{v^2}A_1 - \frac{2}{v_\alpha^3}A_2)}_{N_{x,\alpha}} \underbrace{(v - v_\alpha)}_{\Delta_{x,\alpha}} \quad (44)$$

The fault considered in this application is an actuator fault, that occurs on the actuator. The switched state space representation in this case becomes :

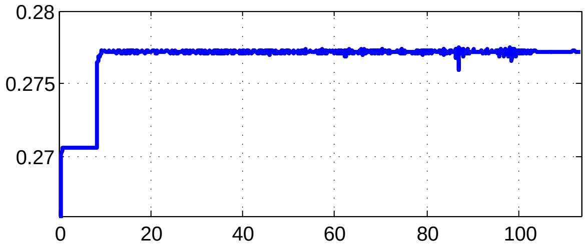

Fig. 3. Steering Angle [rad]

$$\left\{\begin{array}{rcl} \dot{x} & = & (A_\alpha + \Delta_{x,\alpha} N_{x,\alpha})x \\ & & +B_\alpha(u+f) + E_{d,\alpha} d \\ y & = & Cx + D(u+f) \end{array}\right. \quad (45)$$

Applying the LMIs detailed in section IV, a robust fault detection observer can be designed, meeting the 3-objectives detailed before.

The switching ponderation coefficient p in (39) should meet the following constraints:

$$p_\alpha|_{v=v_\alpha} = 0 \text{ and } p_\alpha|_{v=v_{\alpha+1}} = 1$$

Then a suitable p is calculated in this case as follow:

$$p_\alpha = \frac{\frac{1}{v_\alpha} - \frac{1}{v}}{\frac{1}{v_\alpha} - \frac{1}{v_{\alpha+1}}} = (1 - \frac{v_\alpha}{v})(\alpha + 1),$$

For $v \in [v_\alpha, v_{\alpha+1}]$ and $\alpha \in \{1, 2, 3, .., N\}$

The result of such observer is given in figure 4. For a fault that occurs between $t = 23$ and $t = 28$ s, the residuals raises alarming a fault detection. The dashed red curves is for the residual without the weighting, and the full lined blue curve is with the weighting coefficient.

VI. Conclusion and further work

The technique presented in this paper provides a framework for generating a class of robust fault detection observers.

Several time- and frequency-domain specifications have been expressed as LMI constraints on the observers state-space matrices. These analysis are then used for multi-objective synthesis purposes. A compromise of these objectives is proposed as a criterion to minimize. It is formulated an LMIs feasibility problem. The solution of the optimization problem can be found by using efficient LMI solver. An example is given to validate this approch.

In future work, this design will be applied to critical situation detection and sensor faults in lateral vehicle dynamics. The ideas presented here can be generalized for just proper and strictly proper systems, where the same algorithms can be applied to the augmented system.

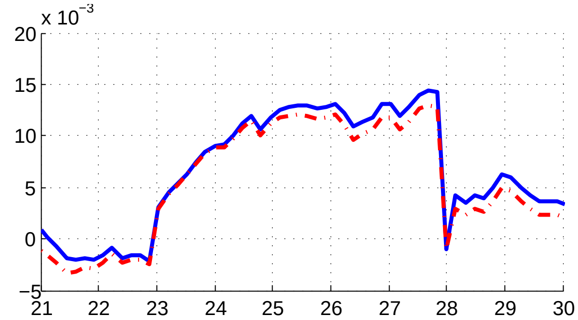

Fig. 4. Residuals signal

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